

Mind Generation
Centru de Matematica si Informatica

Expresii algebrice
Clasa a VIII-a

1. Se considera expresia: $E(x) = \left(\frac{x^2 + 5x + 6}{x^2 - 3x - 10} + 2 \right) \cdot \frac{x^2 - 25}{9x^2 - 42x + 49}$ unde $x \in \mathbb{R} \setminus \left\{ \frac{7}{3}; -2; 5 \right\}$

a). Arătați ca: $E(x) = \frac{x+5}{3x-7}$

b). Determinați $x \in \mathbb{Z}$ pentru care $E(x) \in \mathbb{Z}$

Rezolvare: ajutăm separat expresiile de grad. 2 pt. a le descompune.

$$x^2 + 5x + 6 = \underline{x^2} + \underline{2x} + \underline{3x} + \underline{6} = x(x+2) + 3(x+2) = (x+2)(x+3)$$

Sau pt. cine a învățat ec. de grad. al 2-lea: atașăm ecuația:

$$x^2 + 5x + 6 = 0; \quad a=1, b=5, c=6$$

$$\Delta = b^2 - 4ac = 25 - 4 \cdot 6 = 25 - 24 = 1 > 0$$

$$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} \Rightarrow x_1 = \frac{-5+1}{2} = \frac{-4}{2} = -2$$

$$x_2 = \frac{-5-1}{2} = \frac{-6}{2} = -3$$

} \Rightarrow folosim
descompunerea
 $a(x-x_1)(x-x_2) \Rightarrow$

$$\Rightarrow x^2 + 5x + 6 = [x - (-2)][x - (-3)] = (x+2)(x+3)$$

Bineînțeles că ajungem la același rezultat.

Obs: O expresie de gradul al doilea, se poate descompune intotdeauna folosind aceasta metoda, prin atasarea ecuatiei de gradul 2, aflarea radacinilor si folosirea formulei $a(x-x_1)(x-x_2)$.

Aceasta metoda este sigura intotdeauna si deci se recomanda atunci cand nu ne vin rapid alte idei de decompunere !

$$\text{Mai departe : } x^2 - 3x - 10 = \underline{x^2 - 5x} + 2x - 10 = x(x-5) + 2(x-5) = (x-5)(x+2)$$

$\downarrow \quad \downarrow$
 $-5+2 \quad -5 \cdot 2$

Sau: daca nu vedem gruparea de mai sus: $x^2 - 3x - 10 = 0$

$$\Delta = 3^2 - 4 \cdot (-10) = 9 + 40 = 49 \Rightarrow$$

$$\Rightarrow x_1 = \frac{3 + \sqrt{49}}{2} = \frac{3 + 7}{2} = 5$$

$$x_2 = \frac{3 - \sqrt{49}}{2} = \frac{3 - 7}{2} = -2$$

$$\left. \begin{array}{l} \Rightarrow x_1 = 5 \\ \Rightarrow x_2 = -2 \end{array} \right\} \Rightarrow x^2 - 3x - 10 = (x-5)(x+2)$$

$$x^2 - 25 = x^2 - 5^2 = (x-5)(x+5)$$

$$9x^2 - 42x + 49 = (3x)^2 - 2 \cdot 3x \cdot 7 + 7^2 = (3x-7)^2$$

$(\overset{1}{a^2} \quad -2ab \quad + \overset{2}{b^2})$

$$\text{Deci } E(x) = \left(\frac{(x+2)(x+3)}{(x-5)(x+2)} + 2 \right) \cdot \frac{(x-5)(x+5)}{(3x-7)^2} = \left(\frac{x+3}{x-5} + 2 \right) \cdot \frac{(x-5)(x+5)}{(3x-7)^2} =$$

$x \neq 5 \quad x \neq -2 \quad x \neq \frac{7}{3}$

$$= \frac{x+3+2(x-5)}{\cancel{x-5}} \cdot \frac{\cancel{(x-5)}(x+5)}{(3x-7)^2} = \frac{(x+3+2x-10)(x+5)}{(3x-7)^2} = \frac{\cancel{(3x-7)}(x+5)}{(3x-7)^2} = \frac{x+5}{3x-7} \text{ (c.c.t.d.)}$$

$x \neq 5$ $x \neq \frac{7}{3}$

$$e) \quad \frac{x+5}{3x-7} \in \mathbb{Z} \Rightarrow \left. \begin{array}{l} 3x-7 \mid x+5 \mid \cdot 3 \Rightarrow 3x-7 \mid 3(x+5) \Rightarrow 3x-7 \mid 3x+15 \quad (1) \\ \text{dar } 3x-7 \mid 3x-7 \quad (2) \end{array} \right\} \Rightarrow$$

$$\Rightarrow 3x-7 \mid (3x+15) - (3x-7) \Rightarrow 3x-7 \mid 3x+15-3x+7 \Rightarrow$$
$$\Rightarrow 3x-7 \mid 22 \Rightarrow 3x-7 \in D_{22} \left. \begin{array}{l} \\ \\ \end{array} \right\} \Rightarrow$$
$$D_{22} = \{ \pm 1; \pm 2; \pm 11; \pm 22 \}$$

$$\Rightarrow 3x-7 \in \{ -22; -11; -2; -1; 1; 2; 11; 22 \} \mid +7 \Rightarrow$$

$$\rightarrow 3x \in \{ -15; -4; 5; 6; 8; 9; 18; 29 \} \mid : 3 \Rightarrow$$

$$\Rightarrow x \in \{ -3; -\frac{4}{3}; \frac{5}{3}; 2; \frac{8}{3}; 3; 6; \frac{29}{3} \}$$

dar cum se cere $x \in \mathbb{Z} \Rightarrow$ trebuie să intersectăm cu \mathbb{Z} :

$$x \in \left\{ -3; -\frac{4}{3}; \frac{5}{3}; 2; \frac{8}{3}; 3; 6; \frac{29}{3} \right\} \cap \mathbb{Z} = \{ -3; 2; 3; 6 \} \Rightarrow$$

$$\Rightarrow \underline{\underline{x \in \{ -3; 2; 3; 6 \}}}$$

2. Fie expresia: $\left(\frac{a}{3a-7} - \frac{a}{3a+7} - \frac{7}{9a^2-49} \right)^{-1} : \frac{3a-7}{7}$

Aflați domeniul de definiție al expresiei și valoarea lui a pentru care $E(a)=2$

Rezolvare:

Domeniul de definiție:

1. Tot ce apare la numitor - condiția $\neq 0 \Rightarrow$

$$3a-7 \neq 0 \Rightarrow a \neq \frac{7}{3} \quad (1)$$
$$3a+7 \neq 0 \Rightarrow a \neq -\frac{7}{3} \quad (2)$$
$$9a^2-49 \neq 0 \Rightarrow (3a-7)(3a+7) \neq 0$$
$$\Rightarrow a \neq \pm \frac{7}{3} \quad (3)$$

2. Dacă în expresie avem $^{-1}$ \Rightarrow condiția împărțitorul $\neq 0 \Rightarrow$

$$\Rightarrow \frac{3a-7}{7} \neq 0 \Rightarrow 3a-7 \neq 0 \Rightarrow a \neq \frac{7}{3} \quad (4)$$

3. Dacă avem puteri negative \Rightarrow condiția expresia dinăuntrul puterii negative $\neq 0$ (pt. că va ajunge la numitor!)

$$\begin{aligned} \text{Deci: } \frac{a}{3a-7} - \frac{a}{3a+7} - \frac{7}{9a^2-49} \neq 0 &\Rightarrow \frac{\frac{3a+7}{a}}{3a-7} - \frac{\frac{3a+7}{a}}{3a+7} - \frac{7}{(3a-7)(3a+7)} = \\ &= \frac{a(3a+7) - a(3a-7) - 7}{(3a-7)(3a+7)} = \frac{\cancel{3a^2} + 7a - \cancel{3a^2} + 7a - 7}{(3a-7)(3a+7)} = \frac{14a-7}{(3a-7)(3a+7)} = \frac{7(2a-1)}{(3a-7)(3a+7)} \neq 0 \end{aligned}$$

$$\Rightarrow 2a-1 \neq 0 \Rightarrow a \neq \frac{1}{2} \quad (5)$$

$$\text{Deci din (1)+(2)+(3)+(4)+(5)} \Rightarrow a \in \mathbb{R} \setminus \left\{ -\frac{7}{3}; \frac{1}{2}; \frac{7}{3} \right\}$$

$$E(a) = \left[\frac{7(2a-1)}{(3a-7)(3a+7)} \right]^{-1} : \frac{3a-7}{7} = \frac{(3a-7)(3a+7)}{\cancel{7(2a-1)}} \cdot \frac{\cancel{7}^1}{\cancel{3a-7}^1} = \frac{3a+7}{2a-1} \quad \begin{matrix} a \neq \frac{7}{3} \\ a \neq \frac{1}{2} \end{matrix}$$

$$\Rightarrow E(a) = \frac{3a+7}{2a-1} \Rightarrow E(a) = 2 \Leftrightarrow \frac{3a+7}{2a-1} = 2 \Leftrightarrow 3a+7 = 2(2a-1) \Leftrightarrow$$

$$\Leftrightarrow \underline{\underline{3a+7 = 4a-2}} \Leftrightarrow \underline{\underline{7+2 = 4a-3a}} \Leftrightarrow \underline{\underline{a=9}}$$

3. Se considera: $E(x) = \left(\frac{x}{x^2-x} + \frac{x+2}{2+x-2x^2-x^3} + \frac{x^2}{x^2+x} \right) \cdot \left(x - \frac{1}{x} \right)$

unde $x \in \mathbb{R} \setminus \{-2; -1; 0; 1\}$

Rezolvati in R ecuatia $E(x+1) - E(2x-3) - E(2) = E(3)$

Rezolvare Descompunem numitorii, pentru a afla c.m.m.m.c și a putea amplifica.

$$x^2 - x = x(x-1)$$

$$x^2 + x = x(x+1)$$

$$2+x-2x^2-x^3 = 2+x-x^2(2+x) = (2+x)(1-x^2) = (2+x)(1-x)(1+x)$$

$$\Rightarrow E(x) = \left(\frac{x}{\underset{1}{x(x-1)}} + \frac{x+2}{\underset{1}{(2+x)(1+x)(1+x)}} + \frac{x^2}{\underset{1}{x(x+1)}} \right) \cdot \left(x - \frac{1}{x} \right) =$$

$$= \left(\frac{1}{x-1} + \frac{1}{(1-x)(1+x)} + \frac{x}{x+1} \right) \cdot \frac{x^2-1}{x} = \left(\frac{x+1}{x-1} - \frac{1}{(x-1)(x+1)} + \frac{x}{x+1} \right) \cdot \frac{(x-1)(x+1)}{x} =$$

$$= \frac{x+1-1+x^2-x}{(x-1)(x+1)} \cdot \frac{(x-1)(x+1)}{x} = \frac{x^2}{x} = \underline{\underline{x}}$$

Deci $E(x) = x$

Avem de rezolvat: $E(x+1) - E(2x-3) - E(2) = E(3)$

$E(x+1) = x+1$; $E(2) = 2$

$E(2x-3) = 2x-3$; $E(3) = 3$

$$\left. \begin{array}{l} \Rightarrow x+1 - (2x-3) - 2 = 3 \Leftrightarrow \\ \Leftrightarrow x+1 - 2x+3 - 2 = 3 \Leftrightarrow \\ \Leftrightarrow -x+2 = 3 \Leftrightarrow -x = 3-2 \Leftrightarrow \\ \Leftrightarrow -x = 1 \Leftrightarrow x = -1 \end{array} \right\}$$

4. Fie expresia $E(x) = \left(\frac{x^2+2}{x^2+4x+4} - \frac{x}{x+2} \right) : \left(\frac{1}{x-2} - \frac{3}{x^2-4} \right)$

unde $x \in \mathbb{R} \setminus \{-2; 1; 2\}$
 Determinati $x \in \mathbb{Z}$, astfel incat $E(x) \in \mathbb{Z}$.

Rezolvare

Descompunem numitorii: $x^2+4x+4 = x^2+2 \cdot 2x+2^2 = (x+2)^2$
 $x^2-4 = x^2-2^2 = (x-2)(x+2)$ } \Rightarrow

$$\Rightarrow E(x) = \left(\frac{x^2+2}{(x+2)^2} - \frac{\frac{x+2}{x}}{x+2} \right) : \left(\frac{\frac{x+2}{1}}{x-2} - \frac{3}{(x-2)(x+2)} \right) = \frac{x^2+2 - x(x+2)}{(x+2)^2} : \frac{x+2-3}{(x-2)(x+2)} =$$

$$= \frac{x^2 - x^2 - 2x}{(x+2)^2} \cdot \frac{(x-2)(x+2)}{x-1} \stackrel{x \neq -2}{=} \frac{2(1-x)}{x+2} \cdot \frac{x-2}{x-1} \stackrel{x \neq 1}{=} \frac{-2(x-1)}{x+2} \cdot \frac{x-2}{x-1} = \frac{-2(x-2)}{x+2} = \frac{4-2x}{x+2}$$

$$\Rightarrow \text{de det. } x \in \mathbb{Z} \text{ a.t. } \frac{4-2x}{x+2} \in \mathbb{Z}$$

$$\begin{array}{r} \updownarrow \\ x+2 \mid 4-2x \end{array}$$

$$\text{dar: } x+2 \mid x+2 \cdot 1 \Rightarrow x+2 \mid 2x+4 \quad \left. \vphantom{\begin{array}{r} \updownarrow \\ x+2 \mid 4-2x \end{array}} \right\} \Rightarrow x+2 \mid 4-2x+2x+4 \Rightarrow$$

$$\Rightarrow x+2 \mid 8 \Leftrightarrow x+2 \in D_8 \Leftrightarrow x+2 \in \{-8; -4; -2; -1; 1; 2; 4; 8\} \quad | -2 \Rightarrow$$

$$\Rightarrow x \in \{-10; -6; -4; -3; -1; 0; 2; 6\} \cap \mathbb{Z} = \{-10; -6; -4; -3; -1; 0; 2; 6\} \Rightarrow$$

\Rightarrow deci solutia este:

$$x \in \{-10; -6; -4; -3; -1; 0; 2; 6\}$$

pt. că s-a cerut $x \in \mathbb{Z}$!

(Intersecția în final intersecțăm cu mulțimea unde ni s-a cerut să rezolvăm !)

5. Se considera expresia: $E(x) = \frac{x^2 - 9}{x^2 - 5x + 6} : \frac{(x+2)^2 - x - 4}{x^2 - 2x}$, $x \in \mathbb{R} \setminus \{-3; 0; 2; 3\}$

Calculati:
 $P = E(1) \cdot E(2) \cdot \dots \cdot E(2021)$

Rezolvare: Descompunem:

$$x^2 - 5x + 6 = \underline{x^2 - 2x} - \underline{3x + 6} = \underline{x(x-2)} - \underline{3(x+2)} = (x-2)(x-3) \quad (\text{sau cu ec. de grad. 2})$$

$$x^2 - 9 = (x-3)(x+3)$$

$$x^2 - 2x = x(x-2)$$

$$(x+2)^2 - x - 4 = x^2 + 4x + 4 - x - 4 = x^2 + 3x = x(x+3)$$

$$\rightarrow E(x) = \frac{\overset{1}{\cancel{(x-3)}}(x+3)}{\underset{x \neq 3}{\cancel{(x-2)}\cancel{(x-3)}}} : \frac{\overset{1}{\cancel{x}}(x+3)}{\underset{x \neq 0}{\cancel{x}\cancel{(x-2)}}} = \frac{\overset{1}{\cancel{x-3}}}{\underset{1}{\cancel{x-2}}} \cdot \frac{\overset{1}{\cancel{x+2}}}{\underset{1}{\cancel{x+3}}} = \underline{\underline{1}}$$

$$\Rightarrow \left. \begin{array}{l} E(1) = 1 \\ E(2) = 1 \\ \vdots \\ E(2021) = 1 \end{array} \right\} \Rightarrow P = E(1) \cdot E(2) \cdot \dots \cdot E(2021) = \underline{\underline{1 \cdot 1 \cdot \dots \cdot 1 = 1}}$$

6. Fie $E(x) = (x-2)^2 + 2(x-2)(x+1) + (x+1)^2$

a). Aratati ca $E(x) \geq 0$, oricare ar fi $x \in \mathbb{R}$

b). Calculati $E\left(\frac{1}{\sqrt{2}}\right)$ c). Rezolvati in \mathbb{R} ecuatia: $E(x) - E(x+1) - E(2x-1) + E(2x+1) = E(1)$

Rezolvare:

a) $E(x) = \underbrace{(x-2)^2}_{a^2} + 2 \cdot \underbrace{(x-2)}_a \cdot \underbrace{(x+1)}_b + \underbrace{(x+1)^2}_{b^2} =$ Folosim: $a^2 + 2ab + b^2 = (a+b)^2$

$= (x-2+x+1)^2 = (2x-1)^2 \geq 0, \forall x \in \mathbb{R}$, fiind patrat perfect.

b) $E\left(\frac{1}{\sqrt{2}}\right) = \left(2 \cdot \frac{1}{\sqrt{2}} - 1\right)^2 = \left(\frac{\sqrt{2}}{1} - 1\right)^2 = \left(\frac{1\sqrt{2}}{1} - 1\right)^2 = (\sqrt{2} - 1)^2 = (\sqrt{2})^2 - 2 \cdot \sqrt{2} \cdot 1 + 1^2 = 2 - 2\sqrt{2} + 1 = 3 - 2\sqrt{2}$

c) $E(x) = (2x-1)^2 = 4x^2 - 4x + 1$

$E(x+1) = [2(x+1)-1]^2 = (2x+2-1)^2 = (2x+1)^2 = 4x^2 + 4x + 1$

$E(2x-1) = [2(2x-1)-1]^2 = (4x-2-1)^2 = (4x-3)^2 = 16x^2 - 24x + 9$

$E(2x+1) = [2(2x+1)-1]^2 = (4x+2-1)^2 = (4x+1)^2 = 16x^2 + 8x + 1$

$E(1) = (2 \cdot 1 - 1)^2 = 1$

$$\Rightarrow 4x^2 - 4x + 1 - (4x^2 + 4x + 1) - (16x^2 - 24x + 9) + 16x^2 + 8x + 1 = 1 \Leftrightarrow$$

$$\Leftrightarrow \cancel{4x^2} - \cancel{4x} + 1 - \cancel{4x^2} - \cancel{4x} - 1 - \cancel{16x^2} + \underline{24x} - 9 + \cancel{16x^2} + \underline{8x} + 1 = 1 \Leftrightarrow$$

$$\Leftrightarrow 24x = 9 - 1 + 1 \Leftrightarrow 24x = 9 \quad | :24 \Rightarrow x = \frac{9}{24} \stackrel{13}{=} \frac{3}{8} \in \mathbb{R} \Rightarrow$$

$$\Rightarrow x = \frac{3}{8} \text{ solutie}$$

7. Fie: $E(x) = \frac{x}{x^3 + x^2} : \frac{(x+2)(2x-1) - x(x+3) + 1}{(2x+2)(3x-3)}$, $\forall x \in \mathbb{R} \setminus \{-1; 0; 1\}$

Calculati: $S = E(1) + E(2) + E(3) + \dots + E(2021)$

Rezolvare: $x^3 + x^2 = x^2(x+1)$

$$(x+2)(2x-1) - x(x+3) + 1 = \underline{2x^2} - \underline{x} + \underline{4x} - 2 - \underline{x^2} - \underline{3x} + 1 = x^2 - 1 = (x-1)(x+1) \Rightarrow$$

$$\Rightarrow E(x) = \frac{\cancel{x}}{x^2(x+1)} : \frac{(x-1)(x+1)}{2(x+1) \cdot 3(x-1)} = \frac{1}{x(x+1)} \cdot 6 = \frac{6}{x(x+1)}$$

Folosim egalitatea: $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$, $\forall x \neq 0, x \neq -1$

$$\Rightarrow S(1) = \frac{6}{1 \cdot 2} = 6 \left(\frac{1}{1} - \frac{1}{2} \right)$$

$$S(2) = \frac{6}{2 \cdot 3} = 6 \left(\frac{1}{2} - \frac{1}{3} \right)$$

$$S(3) = \frac{6}{3 \cdot 4} = 6 \left(\frac{1}{3} - \frac{1}{4} \right)$$

:

$$S(2020) = \frac{6}{2020 \cdot 2021} = 6 \left(\frac{1}{2020} - \frac{1}{2021} \right)$$

$$S(2021) = \frac{6}{2021 \cdot 2022} = 6 \left(\frac{1}{2021} - \frac{1}{2022} \right)$$

$$S(1) + S(2) + S(3) + \dots + S(2020) + S(2021) = 6 \left[\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{2020} - \frac{1}{2021} + \frac{1}{2021} - \frac{1}{2022} \right] =$$

$$= 6 \left(\frac{2022}{1} - \frac{1}{2022} \right) = 6 \cdot \frac{2022-1}{2022} = \frac{6 \cdot 2021}{2022} = \frac{2021}{337} \Rightarrow S = \frac{2021}{337}$$

8. Aratati ca numarul $n = (x^2 - x + 2)(x^2 - x + 8) + 9$ este patratul unui numar real, oricare ar fi $x \in \mathbb{R}$

$$n = (x^2 - x + 2)(x^2 - x + 8) + 9 = (x^2 - x)^2 + 2(x^2 - x) + 8(x^2 - x) + 2 \cdot 8 + 9 =$$

$$= (x^2 - x)^2 + 10(x^2 - x) + 25 = (x^2 - x)^2 + 2 \cdot 5 \cdot (x^2 - x) + 5^2 = (x^2 - x + 5)^2 = \text{patrat perfect,} \\ \forall x \in \mathbb{R}$$